WORKSHEET ON EIGENVALUES AND EIGENVECTORS

MATH 186–1

Definition 0.1. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.[1] A non-zero vector $v \in \mathbb{R}^n$ is called a eigenvector for T if there exists a number λ such that $T(v) = \lambda v$. In this case, the number λ is called an eigenvalue for T.

1. Fix $\{u, v\}$ to be a basis for $\mathbb{R}^2$ and fix $\{x, y, z\}$ to be a basis for $\mathbb{R}^3$. Given below are certain vectors and various linear transformations. In each case determine which vectors are eigenvectors and identify the associated eigenvalues.

(a) Set $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation represented by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Try the vectors, u, v, $u + v$, and $u - v$.

u is an eigenvector with associated eigenvalue 2. v is an eigenvector with associated eigenvalue 3. The others are not eigenvectors.

(b) Set $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation represented by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Try the vectors u, v, $u + v$ and $u - v$.

$u + v$ is an eigenvector with associated eigenvalue 1. $u - v$ is an eigenvector with associated eigenvalue -1. The others are not eigenvectors.

(c) Set $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation represented by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Try the vectors u, v, $u + v$ and $u + 2v$.

$u + 2v$ is an eigenvector with associated eigenvalue 2. The others are not eigenvectors.

(d) Set $T : \mathbb{R}^3 \to \mathbb{R}^3$ to be the linear transformation represented by the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$ where a, b are all distinct constants. Try the vectors x, y, z, $x + y$, $3x - 7y$ and $x + y + z$.

$x, y, x + y, 3x - 7y$ are all eigenvectors associated with the eigenvalue a. z is an eigenvector with associated eigenvalue b. $x + y + z$ is not an eigenvector.

[1] We will mostly be concerned with the case that $n = 2$

We'll now begin to develop a better method for identifying eigenvalues and eigenvectors than what we did on the previous page (guess and check). First fix some notation. We will use the letter I to denote the identity linear transformation. That is $I : R^2 \to R^2$ is the map defined by the formula $I(w) = w$ for all $w \in R^2$.

2. Suppose that $T : R^2 \to R^2$ is a linear transformation. Let's suppose that a vector w is an eigenvector for T with associated eigenvalue λ. Prove that the new linear transformation $(\lambda \cdot I - T)$ is not injective. Here $(\lambda \cdot I - T)$ is defined by the rule $(\lambda \cdot I - T)(x) = T(x) - \lambda \cdot I(x)$ for all $x \in R^2$.

I'm going to assume that w is a non-zero vector. Eigenvectors are always assumed to be non-zero (I should have said this more clearly on the first page). Then

$$(\lambda \cdot I - T)(w) = \lambda w - T(w) = \lambda w - \lambda w = 0.$$

But $(\lambda I - T)(0) = 0$ also so $(\lambda I - T)$ sends two different vectors to zero and so it is not injective.

3. With notations in problem #2, fix a basis {u, v} for R^2. Assume that T is represented by the matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, write down a matrix representation of $(\lambda \cdot I - T)$. Finally write down the determinant of this matrix you constructed (note that this determinant is a polynomial in the variable λ, it is called the characteristic polynomial of the matrix).

The matrix representation of $(\lambda \cdot I - T)$ is $\begin{bmatrix} \lambda - e & -f \\ -g & \lambda - h \end{bmatrix}$. The determinant is

$$(\lambda - e)(\lambda - h) + fg$$

Remark 0.2. One can do something similar for 3 × 3 matrices. In particular, there is a determinant of such matrices and you can construct the characteristic polynomial in the same way. Versions of the results on the following pages also hold for 3 × 3 matrices.

4. Suppose that k is a real number. Show that that k is a root of the polynomial from problem 3.

[ifandonlyifkisaneigenvalueforT.

a b ,thenTisinjectiveifandonlyifad−bc6=0.
Hint:In]thehomeworkyouturnedinyesterday,youshowedthataifTwasrepresentedbyamatrix

F[ixaba]sisu,v∈R2andfixalineartransformationT:R2→R2representedbythematrix

ef justasinthepreviousproblem.

Suppose first that k is a root of the characteristic polynomial of this matrix. Then det(k
·I−T) = 0 · I − T) is not injective. But then there exists a non-zero w such that − T)(w) = 0 or
and in particular, (k
(kI · in other words kw = kI · (w) = T(w) which proves that w is an eigenvector

with associated eigenvalue k.

Conversely,supposethat kisaneigenvalueforthematrix [e f].Thenithasanassociated

nonzero eigenvector w. Thus kw = T(w) and it follows (reversing the steps from above) that − T)(w) =
(kI · 0. But then det(kI · − T) = (k − e)(k − h) + fg = 0. Thus k is a root of the

polynomial (λ − e)(λ − h) + fg.

5. Compute t heeigenvaluesofthelineartransformationsfromproblem#1(a),(b),(c).What's
stopping youfrom]computingtheeigenvaluesforthelineartransformationcorrespondingtothe
[matrix 0 1
−10?(Geometrically,remindyourselfwhatthislineartransformationdoes).

(a) The characteristic polynomial is (λ − 2)(λ − 3) and so the roots (and thus eigenvalues) are −
2 and 3.

(b) The characteristic polynomial is λ2 1 and so the roots (and thus eigenvalues) are −1 and − 1)
1

(c) The characteristic polynomial is λ(λ − 2 = λ2 − λ − 2 and so the roots (and thus
eigenvalues) are 2 and −1.

(d) The characteristic polynomial is λ2 + 1. Thus polynomial doesn't have any roots! Geometrically, it corresponds to rotation by 90 degrees (and so geometrically, one would not expect any eigenvectors either).

6. Can a linear transformation $T : R2 \to R2$ have more than 2 distinct eigenvalues? Justify your answer.

No, the eigenvalues of T are always the roots of a polynomial equation of degree 2. Such equations can have at most 2 roots (although sometimes they can also have 1 root or zero roots).

7. Suppose that $T : R2 \to R2$ is a non-surjective linear transformation. Prove that $\lambda = 0$ is an eigenvalue for T.

Since T is non-surjective, it is non-injective. Thus $T(w) = T(w')$ for two distinct vectors w and w'. Then $T(w - w') = T(w) - T(w') = 0 = 0(w - w')$. In particular, 0 is an eigenvalue for the eigenvector $w - w' \neq 0$.

8. Suppose that $T : R2 \to R2$ is a linear transformation. Further suppose that $x, y \in R2$ are linearly independent eigenvectors of T but they have the same eigenvalue λ. Show that every vector in R2 is an eigenvector of T (associated to the same eigenvalue) and also that the characteristic polynomial of the matrix associated to T has a double-root at λ. What would it mean about T if $\lambda = 0$?

Fix any vector $w \in R2$. Since x,y are linearly independent, they are a basis and so we can write $w = ax + by$. But then

$$T(w) = T(ax + by) = aT(x) + bT(y) = a\lambda x + b\lambda y = \lambda(ax + by) = \lambda w$$

as desired.

Now we show that the characteristic polynomial has a double root. We know that it has one root λ and so if we write the characteristic polynomial $z^2 + dz + e$ with the variable z (other letters already seem to be used), then $(z - \lambda)(z - ???) = z^2 + dz + e$ using polynomial long division. Let us use the variable γ instead of ???. Then γ must be an eigenvalue with associated eigenvector $w' \neq 0$. But $w' \in R2$ so w' is also an eigenvector associated to λ. In other words

$$\lambda w' = T(w') = \gamma w'.$$

This implies that $\gamma = \lambda$.

Finally if $\lambda = 0$, then for any $w = R2$, $T(w) = 0w = 0$. In particular, T is the linear transformation that sends all vectors to 0. It is represented by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ no matter what basis you use.

Now we turn to the question of finding the eigenvectors associated to a given eigenvalue. Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation represented by a matrix $\begin{bmatrix} e & f \\ [illegible] \end{bmatrix}$ and that λ is an eigenvalue. To find the eigenvectors associated to λ, write

$$\begin{bmatrix} e & f \\ [illegible] \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now expand the left side of the equation and obtain equations (viewing x and y as variables). Find any pair of x and y that satisfy those equations and you have found an eigenvector. Let us do an explicit example:

Example 0.3. Suppose we are given the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. By the method described above, one can verify that the number 5 is an eigenvalue of the linear transformation associated to A. So we write

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}.$$

The right side of the equation is just $\begin{bmatrix} 5x \\ 5y \end{bmatrix}$ and the left side is $\begin{bmatrix} x+2y \\ 4x+3y \end{bmatrix}$. So we have the equations

$$x + 2y = 5x$$
$$5y = 4x + 3y$$

Which reduces (in either case) to $y = 2x$. Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector (as is $\begin{bmatrix} -7 \\ -14 \end{bmatrix}$).

9. Using this method, find the eigenvectors associated to the matrices from problem #1(a)(b)(c). Also, find the eigenvalues and eigenvectors associated to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. This last one is fairly messy.

(a) All scalar multiples of $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are the eigenvectors for the eigenvalue 2. All scalar multiples of $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the eigenvectors for the eigenvalue 3.

(b) All scalar multiples of $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the eigenvectors for the eigenvalue 1. All scalar multiples of $u - v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigenvectors for the eigenvalue -1.

(c) All scalar multiples of $u + 2v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are the eigenvectors for the eigenvalue 2. All scalar multiples of $-u + v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvectors for the eigenvalue -1.

(d) All scalar multiples of $u + \frac{3+\sqrt{33}}{4} v = \begin{bmatrix} 1 \\ \left(\frac{3+\sqrt{33}}{4}\right) \end{bmatrix}$ are the eigenvectors for the eigenvalue $\frac{5+\sqrt{33}}{2}$. All scalar multiples of $u + \frac{3-\sqrt{33}}{4} v = \begin{bmatrix} 1 \\ \left(\frac{3-\sqrt{33}}{4}\right) \end{bmatrix}$ are the eigenvectors for the eigenvalue $\frac{5-\sqrt{33}}{2}$.